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Note

On the Degree of Approximation by Bernstein Polynomials

The aim of this note is to show that the novel approach taken in [2] to introduce the Bernstein polynomials can be used to study the degree of approximation of differentiable functions by these polynomials. The idea of [2] is to approximate a smooth function f defined on [0, 1] by approximating on a certain closed region the function $U(x, y) \equiv f(x + y)$ satisfying

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y}, \qquad U(x, 0) = f(x).$$

For each psotive integer N and each $t \in (0, 1]$, an approximate solution $u(ih, nk) \equiv u_{i,n}$ is constructed on the grid points (ih, nk) of the set $\{(x, y): 0 \leqslant x, 0 \leqslant y \leqslant t, x + y \leqslant 1\}$ by use of the forward difference scheme

$$u_{i,n+1} = (1 - t)\, u_{i,n} + t u_{i+1,n}, \qquad u_{i,0} = f(i/N), \tag{1}$$

where $h = 1/N$ and $k = th$. It is then easy to see that the Nth order Bernstein polynomial of f at t is given by

$$B_N(f; t) = \sum_{i=0}^{N} \binom{N}{i} (1 - t)^{N-i}\, t^i f(i/N) = u_{0,N}.$$

THEOREM. *If a real function f is continuous on* [0, 1] *and has a bounded derivative on* (0, 1), *then for every $t \in [0, 1]$ and for $N = 1, 2, \ldots$,*

$$| B_N(f; t) - f(t)| \leqslant t\omega(f'; 1/N),$$

where $\omega(f'; \cdot)$ is the modulus of continuity of f' on (0, 1).

Proof. For $0 \leqslant n \leqslant N$ and $0 \leqslant i \leqslant N - n$ we set

$$T_{i,n} = U(ih, (n + 1)k) - U(ih, nk) - t[U((i + 1)h, nk) - U(ih, nk)]. \tag{2}$$

By use of the mean value theorem and the fact that $U(x, y) \equiv f(x + y)$ we have

$$T_{i,n} = kf'(\theta_1) - thf'(\theta_2),$$

where $\theta_1, \theta_2 \in (ih + nk, (i + 1)h + nk)$. Therefore

$$| T_{i,n} | \leqslant t\omega(f'; 1/N)/N. \tag{3}$$

If $T_N = \max\{| T_{i,n} |: 0 \leqslant n \leqslant N, 0 \leqslant i \leqslant N - n\}$, then by use of (1) and (2) it follows easily as in [2] that

$$| f(t) - B_N(f; t)| = | U(0, Nk) - u_{0,N} | \leqslant NT_N .$$

The desired result then follows from (3).

Lorentz [3, p. 21] obtained the estimate[1]

$$| f(t) - B_N(f; t)| \leqslant \tfrac{3}{4}\omega(f'; 1/\sqrt{N})/\sqrt{N} \qquad (N = 1, 2,...). \tag{4}$$

Since $\omega(f'; 1/\sqrt{N}) \leqslant (1 + \sqrt{N})\, \omega(f'; 1/N)$, it follows that

$$\overline{\lim_{N\to\infty}} \frac{\omega(f'; 1/\sqrt{N})/\sqrt{N}}{\omega(f'; 1/N)} \leqslant 1$$

and hence for a given $t \in (0, 1]$ Lorentz's estimate is essentially at least as good as ours. In fact, if $f(t) \equiv t^{\alpha+1}$, where $0 < \alpha < 1$, then one can see that Lorentz's result is strictly better. However, if f' satisfies a Lipshitz condition of order one on (0, 1), then for each $t \in [0, 1]$ both estimates are $O(1/N)$ which cannot be improved to $o(1/N)$ unless f is linear in view of a theorem of Bajšanski and Bojanić [1].

References

1. B. Bajšanski and R. Bojanić, A note on approximation by Bernstein polynomials, *Bull. Amer. Math. Soc.* **70** (1964), 675–677.
2. C. W. Groetsch and J. T. King, The Bernstein polynomials and finite differences, *Math. Mag.* **46** (1973), 280–282.
3. G. G. Lorentz, "Bernstein Polynomials," University of Toronto Press, Toronto, 1953.

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[1] Lorentz works in a slightly different setting, but his proof yields (4) in our setting.